# Many-player entangled state solutions in game theory problems

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We propose a non-classical multi-player entangled state which eliminates the need for communication, yet can solve problems (that require coordination) better than classical approaches. For the entangled state, we propose a slater determinant of all allowed states of a filled band in a condensed matter system – the integer quantum Hall state at filling factor 1. Such a state gives the best solution (i.e., best Nash equilibrium) for some classical stochastic problems where classical solutions are far from ideal.

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## I. INTRODUCTION

Game theoretical problems dealing with conflict of interest have been tackled in the recent past with quantum approaches[1]. It is hoped that quantum game theory, by exploiting quantum mechanics, would produce significantly improved solutions. Of particular interest is how many-particle quantum entanglement can be harnessed to provide better strategies (in games) compared to the classical solutions. Entanglement, which provides correlations between remote particles, can equip the players with a coordinated set of actions depending on the state of the particle that they privately observe. Thus when players cannot communicate by classical channels, they can still arrive at an optimal strategy.

Quantum non-locality was first demonstrated by Bell with his famous inequalities[2]. Quantum theory predicts correlations among outcomes of distant measurements which cannot be explained using only local variables. It has been demonstrated that two photons are correlated over large distances (of the order of 10 km) thereby violating Bell's inequalities[3]. Thus we have a verification of the basic assumption of quantum information and computation that quantum systems can be entangled over large distances and times.

In the past quantum entanglement has been incorporated in classical two-party games such as the prisoner's dilemma by Eisert *et al.*[4], the battle of sexes by Marinatto and Weber[5], etc. These authors demonstrated how optimal solutions can be achieved using entanglement. The purpose of the present work is to propose many-particle entangled states and show how they can be used to obtain improved/optimal solutions for classical problems requiring coordinated action by the players.

## II. MANY-PARTICLE ENTANGLED STATE

In condensed matter systems one frequently encounters bands filled with fermions. Based on Pauli's exclusion principle, the ground state of any $N$-state band filled completely by $N$ spinless-fermions is a Slater determinant of the complete set of $N$ single particle eigen states of the band. Such a Slater determinant is an antisymmetric linear superposition of $N!$-many $N$-particle eigen states. Here *we consider a case of a degenerate filled band – the integer quantum Hall state at filling factor 1.*

In our quantum Hall state, electrons are chosen to be confined to the xy-plane and subjected to a perpendicular magnetic field. On choosing a symmetric gauge vector potential, $\vec{A} = 0.5B(y\widehat{x} - x\widehat{y})$, the degenerate single-particle wavefunctions for the lowest Landau level (LLL) are given by:

$$\phi_m(z) \equiv |m\rangle = \frac{1}{(2\pi l_0^2 2^m m!)^{\frac{1}{2}}} \left(\frac{z}{l_0}\right)^m e^{-|z|^2/4l_0^2}, \quad (1)$$

where $z = x - iy$ is the electron position in complex plane, $m$ is the orbital angular momentum, and $l_0 \equiv \sqrt{\hbar c/eB}$ is the magnetic length. The area occupied by the electron in state $|m\rangle$ is

$$\langle m|\pi r^2|m\rangle = 2(m+1)\pi l_0^2. \quad (2)$$

Thus the LLL can accommodate only $N_e$ electrons given by

$$N_e = (M+1) = \frac{A}{2\pi l_0^2}, \quad (3)$$

where $A$ is the area of the system and $M$ is the largest allowed angular momentum for area $A$. The many-electron system is described by the Hamiltonian

$$H = \sum_j \frac{1}{2m_e}\left[-i\hbar\nabla_j - \frac{e}{c}\vec{A}_j\right]^2 + \sum_{j<k}\frac{e^2}{|z_j - z_k|} + g\mu_B \sum_j \vec{B}\cdot\vec{s}_j. \quad (4)$$

Thus when the LLL (with the lowest Zeeman energy) is completely filled with $N_e$ electrons (i.e., when LLL is at filling factor $\nu = 1$), the many-particle wavefunction $\Psi(z_1, z_2, ...., z_{N_e})$ is given by the Slater determinant

$$\begin{vmatrix} \phi_0(z_1) & \phi_0(z_2) & \ldots & \phi_0(z_{N_e}) \\ \phi_1(z_1) & \phi_1(z_2) & \ldots & \phi_1(z_{N_e}) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{N_e-1}(z_1) & \phi_{N_e-1}(z_2) & \ldots & \phi_{N_e-1}(z_{N_e}) \end{vmatrix}. \quad (5)$$

The many-particle wavefunction $\Psi(z_1, z_2, ...., z_{N_e})$ for $N_e$ particles can be expressed as follows:

$$\Psi(z_1, z_2, ...., z_{N_e}) = \psi(z_1, z_2, ...., z_{N_e}) e^{-\sum_{l=1}^{N_e} |z_l|^2/4l_0^2}, \quad (6)$$

where

$$\begin{aligned}\psi(z_1, z_2, ...., z_{N_e}) &= \prod_{1\le j<k\le N_e} (z_j - z_k) \\ &= \sum_{\sigma \in S_{N_e}} \mathrm{sgn}(\sigma) z_1^{\sigma(1)-1} ... z_{N_e}^{\sigma(N_e)-1}, \quad (7)\end{aligned}$$

where $S_{N_e}$ denotes the set of permutations of $\{1, 2, ..., N_e\}$ and $\mathrm{sgn}(\sigma)$ denotes the signature of the permutation $\sigma$. Thus we see that $\psi(z_1, z_2, ...., z_{N_e})$ is a linear superposition of $N_e!$ states (all with the same probability of being observed) and each state $z_1^{\sigma(1)-1} ... z_{N_e}^{\sigma(N_e)-1}$ has the angular momenta $0, 1, 2, ..., N_e - 1$ distributed among $N_e$ fermionic particles (at positions $z_1, z_2, ..., z_{N_e}$) in a uniquely different way (with no two particles having the same angular momentum)! Thus if the many-particle wavefunction $\Psi(z_1, z_2, ...., z_{N_e})$ is measured for angular momentum of its particles (using for instance a Stern-Gerlach type of set-up) at positions $z_1, z_2, ..., z_{N_e}$, then one of the $N_e!$ permutations of the angular momentum from the set $\{0, 1, 2, ..., N_e - 1\}$ will be measured with probability $1/(N_e!)$. The above fact can be exploited in a game-theoretic context as described in the next section.

Here it should be pointed out that although an antisymmetric wavefunction obtained based on Pauli's exclusion principle is in general not an entangled state[6,7], the Coulomb interactions actually produce the same antisymmetric wavefunction even when the fermionic nature of the particles is ignored, i.e., for example if the particles are treated as classical particles. Furthermore, for the situation where the g-factor is zero (which can be achieved in gallium arsenide heterostructures using pressure), Coulomb interaction energy is minimized when the real space wave function is antisymmetric and given by Eq. (6) while the spin wavefunction is symmetric (with the total spin being maximized and equal to $N_e/2$). This is clearly an entangled state based on correlation effects. This situation is very similar to that of the electronic wavefunction in a half-filled degenerate sub-shell in an atom (such as the five electrons in the 3d sub-shell of $Mn^{2+}$) where Hund's rule dictates that wavefunction be antisymmetric in the real space and symmetric in the spin space. In general, for the quantum Hall situation (at filling factor 1) where one has at least two species of fermionic particles with all the particles having the same charge, spin, and single particle energy $(\hbar\omega_c/2 - 0.5 g\mu_B B)$, one again obtains [for total number of particles $N = N_e = A/(2\pi l_0^2)$] the same many-body wavefunction [given by Eq. (6)] which now is certainly entangled due to correlation effects produced by Coulomb interactions. Lastly, we would like to add that, the above considerations for minimum Coulomb interaction energy are certainly valid when the repulsive interaction is given by a short range Dirac-delta function in which case the interaction energy is zero.

## III. QUANTUM SOLUTIONS TO CLASSICAL PROBLEMS

In this section we will pose a couple of classical problems and show that entanglement not only significantly improves the solution, in fact, it also produces the best possible solution.

### A. Kolkata restaurant problem

We will first examine the Kolkata restaurant problem (KRP)[8] which is a variant of the Minority Game Problem[9]. In the KRP (in its minimal form) there are $N$ restaurants (with $N \to \infty$) that can each accommodate only one person and there are $N$ agents to be accommodated. All the $N$ agents take a stochastic strategy that is independent of each other. If we assume that, on any day, each of the $N$ agents chooses randomly any of the $N$ restaurants such that if $m$ $(> 1)$ agents show up at any restaurant, then only one of them (picked randomly) will be served and the rest $m - 1$ go without a meal. It is also understood that each agent can choose only one restaurant and no more. Then the probability $f$ that a person gets a meal (or a restaurant gets a customer) on any day is calculated based on the probability $P(m)$ that any restaurant gets chosen by $m$ agents with

$$P(m) = \frac{N!}{(N-m)!m!} p^m (1-p)^{N-m} = \frac{\exp(-1)}{m!}, \quad (8)$$

where $p = 1/N$ is the probability of choosing any restaurant. Hence, the fraction of restaurants that get chosen on any day is given by

$$f = 1 - P(0) = 1 - \exp(-1) \approx 0.63. \quad (9)$$

Now, we extend the above minimal KRP game to get a more efficient utilization of restaurants by taking advantage of past experience of the diners. We stipulate that the successful diners $(NF_n)$ on the $n$th day will visit the same restaurant on all subsequent days as well, while the remaining $N - NF_n$ unsuccessful agents of the $n$th day try stochastically any of the $N$ restaurants on the next day (i.e., $n+1$th day) and so on until all customers find a restaurant. The above procedure can be mathematically modeled to yield the following recurrence relation

$$F_{n+1} = F_n + f(1 - F_n)^2, \quad (10)$$

where $F_n$ is the fraction of restaurants occupied on the $n$th day with $F_1 = f = 1 - 1/e$. Upon making a continuum approximation we get

$$\frac{dF}{dn} = f(1 - F)^2, \quad (11)$$

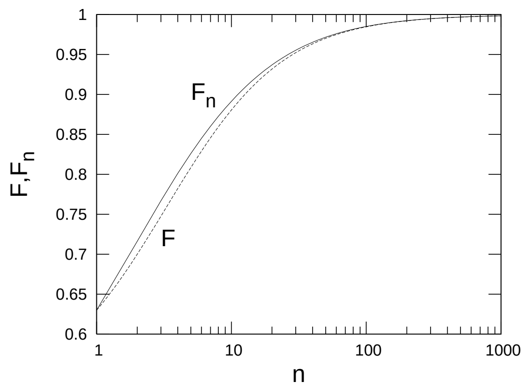


FIG. 1: Plot of the exact probability $F_n$ and the continuum approximation probability $F$ for various iteration values $n$.

which yields the solution

$$F = 1 - \frac{e}{e^2 + (n-1)(e-1)}. \qquad (12)$$

The above solution $F$ turns out to be a good approximation to the solution for $F_n$ in Eq. (10) (with error less than 5%) as can be seen from Fig. 1. We see that even after 10 iterations less than 90% of the restaurants are filled!

We will now investigate how superior quantum solutions can be obtained for the KRP. We will introduce quantum mechanics into the problem by asking the $N$ agents to share an entangled $N$-particle quantum Hall state at filling factor 1 described in the previous section [see Eqs. (6) & (7)]. We assign to each of the $N$ restaurants a unique angular momentum picked from the set $\{0, 1, 2, ..., N-1\}$. We ask each agent to measure the angular momentum of a randomly chosen particle from the $N$-particle entangled state. Then, based on the measured angular momentum, the agent goes to the restaurant that has his/her particular angular momentum assigned to it. In this approach all the agents get to eat in a restaurant and all the restaurants get a customer. Thus we see that the prescribed entangled state always produces restaurant-occupation probability 1 and is thus superior to the classical solution mentioned above! Furthermore, the probability that an agent picks a restaurant is still $p = 1/N$ and hence all agents are equally likely to go to any restaurant. Thus, even if there is an accepted-by-all hierarchy amongst the restaurants (in terms of quality of food with price of all restaurants being the same), the entangled state produces an equitable (Pareto optimal) solution where all agents have the same probability of going to the best restaurant, or the second-best restaurant, and so on. *Quite importantly, it can be shown that the chosen entangled quantum strategy (i.e., the entangled $N$-particle quantum Hall state at filling factor 1) actually represents the best Nash equilibrium when there is a restaurant ranking!* (see Appendix A for details).

TABLE I: The calculated values of the cumulated probability $P$ for a system with $NK$ persons and $K$ gates with a gate-capacity $\alpha N$.

| $\alpha$ | $N$ | $K$ | $P$ | | $\alpha$ | $N$ | $K$ | $P$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 100 | 10 | 0.5266 | | 1.05 | 100 | 10 | 0.7221 |
| 1 | 500 | 10 | 0.5119 | | 1.05 | 500 | 10 | 0.8848 |
| 1 | 1000 | 10 | 0.5084 | | 1.05 | 1000 | 10 | 0.9531 |
| 1 | 10000 | 10 | 0.5027 | | 1.05 | 10000 | 10 | 1.0000 |
| 1 | 100 | 20 | 0.5266 | | 1.1 | 100 | 10 | 0.8652 |
| 1 | 500 | 20 | 0.5119 | | 1.1 | 500 | 10 | 0.9907 |
| 1 | 1000 | 20 | 0.5084 | | 1.1 | 1000 | 10 | 0.9995 |
| 1 | 10000 | 20 | 0.5027 | | 1.1 | 10000 | 10 | 1.0000 |

### B. Kolkata stadium problem

We will next introduce a variant of the KRP game which we will call as the Kolkata stadium problem (KSP). In the KSP, there are $NK$ agents trapped inside a Kolkata stadium that has $K$ exits. There is a panic situation of a fire or a bomb-scare and all the agents have to get out quickly through the $K$ exits each of which has a capacity of $\alpha N$ with $\alpha \geq 1$. We assume that all $NK$ agents have equal access to all the exits and that each agent has enough time to approach only one exit before being harmed. The probability $P(m)$ that any exit gets chosen by $m$ agents is given by the binomial distribution

$$P(m) = \frac{(NK)!}{(NK-m)!m!} p^m (1-p)^{NK-m}, \qquad (13)$$

where $p = 1/K$ is the probability of choosing any gate. For a capacity of $\alpha N$ for each gate, the cumulative probability $P = \sum_{m=1}^{\alpha N} P(m)$ that (on an average) a gate is approached by $\alpha N$ or less agents is given in Table I.

Thus we see that if a gate has the optimal capacity of $N$ (i.e., $\alpha = 1$), then $P$ is close to 0.5 and is not affected by the number of gates $K$ (for small $K$) with $P \to 0.5$ for $N \to \infty$. However, as $\alpha$ increases even slightly above unity, $P$ increases significantly for fixed values $N$ and $K$. Furthermore, for fixed values of $\alpha > 1$ and $K$ (with $\alpha$ only slightly larger than 1 and $K$ being small) $P \to 1$ as $N$ becomes large. *Here it should be mentioned that even when $P \to 1$ on an average, there can be fluctuations in a stampede situation with more than $\alpha N$ people approaching a gate and thus resulting in fatalities.*

Here too in the KSP game, if the $NK$ agents were to use the entangled $NK$-particle state given by the quantum Hall effect state at filling factor 1, then every agent is assured of safe passage. In this situation, since there are $NK$ angular momenta and only $K$ gates, the angular momentum $M_i$ measured by an agent $i$ for his/her particle should be divided by $K$ and the remainder be taken to give the appropriate gate number [i.e., gate number = $M_i$ (**mod** $K$)]. Thus entanglement gives safe exit with

probability 1 even when $\alpha = 1$! Furthermore, even if there is an accepted-by-all ranking of the exits in terms of comfort of passage, our chosen entangled state corresponds to the best Nash equilibrium!

## IV. CONCLUSIONS

In the $N$-agent KRP game, while the number of satisfactory choices is only $N!$, in sharp contrast the number of possibilities is $N^N$ when all the restaurants have the same ranking. Thus, in the classical stochastic approach, the probability of getting the best solution where all the restaurants are occupied by one customer is given by the vanishingly small value $\exp(-N)$. Even in the KSP case, it can be shown that there is a vanishingly small probability $[=\sqrt{K/(2\pi N)^{K-1}}]$ of providing safe passage to all when only $N$ people are allowed to exit from each of the $K$ gates (i.e., when $\alpha = 1$). On the other hand, in this work we showed how quantum entanglement can produce a coordinated action amongst all the $N$-agents leading to the best possible solution with a probability 1!. Thus quantum entanglement produces a much more desirable scenario compared to a classical approach at least for the KRP and the KSP games.

As a candidate for entanglement we could have picked any filled band system (of condensed matter physics) even in the absence of a magnetic field. For such an entangled $N$-particle state, momentum is a good quantum number. However, only when the Coulomb interaction is infinity compared to the kinetic energy do we have the many-body ground state given by the antisymmetric wavefunction satisfying Pauli's exclusion principle. Furthermore, the minimum spacing between various particle momenta is only $2\pi\hbar/L$ where $L$ is the linear size of the system and hence, to unambiguously determine the momentum of a particle, one is faced with the uncertainty principle which fixes the uncertainty in the measured momentum to be at least $\hbar/2L$.

Next, one can also consider $N$ number of identical qudits each with $N$ possible states. By producing an antisymmetric entangled state from these N qudits, one can get better results than classical approaches. However, physically realizing a qudit with a large number of states is a challenging task[11].

Lastly, although it has not been shown that our many-particle entangled state (i.e., the quantum Hall effect state at filling factor 1) will have long-distance and also long-term correlations, we are hopeful of such a demonstration in the future.

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TABLE II:

| | $R_1$ | $R_2$ |
|---|---|---|
| $R_1$ | $(\frac{u_1}{2}, \frac{u_1}{2})$ | $(u_1, u_2)$ |
| $R_2$ | $(u_2, u_1)$ | $(\frac{u_2}{2}, \frac{u_2}{2})$ |

## APPENDIX A

In a $N$-player game, the set of strategies $(s_1^*, s_2^*, ..., s_N^*)$ represent a *Nash Equilibrium* if, for every player $i$, the strategy $s_i^*$ meets the following requirement for the payoff function \$:

$$\begin{aligned} &\$_i(s_1^*, ...., s_{i-1}^*, s_i^*, s_{i+1}^*, ...., s_N^*) \\ &\quad \geq \$_i(s_1^*, ...., s_{i-1}^*, s_i, s_{i+1}^*, ...., s_N^*), \end{aligned} \tag{A1}$$

for every strategy $s_i$ available to $i$. In order to illustrate the main idea behind exploiting quantum strategies, we will consider the simple situation of two restaurants $R_1$ and $R_2$ with utility $u_1$ and $u_2$ respectively as perceived by two diners $D_1$ and $D_2$. Then we can represent the payoff for the diners by using the bimatrix displayed in Table II with diner $D_1$ choices along the rows and those of $D_2$ along the columns.

Here we use the formalism developed in Ref. 5. We assume that diners $D_{1,2}$ have access to the following entangled state:

$$|\psi_{in}\rangle = a|R_1R_2\rangle + b|R_2R_1\rangle, \tag{A2}$$

where the coefficients satisfy the condition $|a|^2 + |b|^2 = 1$. The corresponding initial density matrix is given by

$$\rho_{in} = \rho_{in}^{D_1} \otimes \rho_{in}^{D_2} = |\psi_{in}\rangle\langle\psi_{in}|. \tag{A3}$$

We assume that each player can manipulate his state (i.e., restaurant) by either using the identity $I$ or the Pauli flipping operator $\sigma_x$ which is unitary and has the following property

$$\sigma_x|R_{1,2}\rangle = |R_{2,1}\rangle. \tag{A4}$$

We further assume that each diner can transform his part $(\rho_{in}^{D_{1,2}})$ of the total density matrix $\rho_{in}$ in the following manner:

$$\rho_{fin}^{D_{1,2}} = p_{1,2} I \rho_{in}^{D_{1,2}} I^\dagger + (1 - p_{1,2})\sigma_x \rho_{in}^{D_{1,2}} \sigma_x^\dagger, \tag{A5}$$

to obtain the final density matrix

$$\rho_{fin} = \rho_{fin}^{D_1} \otimes \rho_{fin}^{D_2}. \tag{A6}$$

In order to evaluate the payoff, we define the payoff operator as follows:

$$\begin{aligned} P_{1,2} = &\, u_{1,2}|R_1R_2\rangle\langle R_1R_2| + u_{2,1}|R_2R_1\rangle\langle R_2R_1| \\ &+ 0.5u_1|R_1R_1\rangle\langle R_1R_1| + 0.5u_2|R_2R_2\rangle\langle R_2R_2|. \end{aligned} \tag{A7}$$

Then the payoffs obtained using the following expression

$$\$_{1,2} = \mathrm{Tr}(P_{1,2}\rho_{fin}), \tag{A8}$$

are given by

$$\begin{aligned}\$_1(p_1,p_2) = & \, 0.5p_1p_2(u_1+u_2) \\ & +p_1\left[0.5u_1|a|^2+0.5u_2|b|^2-u_2|a|^2-u_1|b|^2\right] \\ & -0.5p_2\left[u_1|b|^2+u_2|a|^2\right] \\ & +u_2|a|^2+u_1|b|^2, \end{aligned} \tag{A9}$$

and

$$\begin{aligned}\$_2(p_1,p_2) = & \, 0.5p_1p_2(u_1+u_2) \\ & -0.5p_1\left[u_1|a|^2+u_2|b|^2\right] \\ & +p_2\left[0.5u_1|b|^2+0.5u_2|a|^2-u_1|a|^2-u_2|b|^2\right] \\ & +u_1|a|^2+u_2|b|^2. \end{aligned} \tag{A10}$$

To determine the Nash equilibria, we stipulate that the following differences be non-negative:

$$\begin{aligned}\$_1(p_1^*,p_2^*)-\$_1(p_1,p_2^*) = (p_1^*-p_1) & \left[0.5p_2^*(u_1+u_2)\right. \\ & +u_1(0.5|a|^2-|b|^2) \\ & \left.-u_2(|a|^2-0.5|b|^2)\right], \end{aligned} \tag{A11}$$

and

$$\begin{aligned}\$_2(p_1^*,p_2^*)-\$_2(p_1,p_2^*) = (p_2^*-p_2) & \left[0.5p_1^*(u_1+u_2)\right. \\ & +u_2(0.5|a|^2-|b|^2) \\ & \left.-u_1(|a|^2-0.5|b|^2)\right]. \end{aligned} \tag{A12}$$

Then, from Eqs. (A11) and (A12), we obtain the three Nash equilibria $(p_1,p_2)=(1,1),(0,0)$, and $(\bar{p}_1,\bar{p}_2)$ where

$$\bar{p}_1 \equiv -[u_1(1-3|a|^2)+u_2(-2+3|a|^2)]/(u_1+u_2), \tag{A13}$$

and

$$\bar{p}_2 \equiv -[u_1(-2+3|a|^2)+u_2(1-3|a|^2)]/(u_1+u_2). \tag{A14}$$

Next, we note that the differences

$$\$_1(1,1)-\$_1(0,0)=(u_2-u_1)(1-2|a|^2), \tag{A15}$$

and

$$\$_2(1,1)-\$_2(0,0)=-(u_2-u_1)(1-2|a|^2), \tag{A16}$$

are equal in magnitude but opposite in sign. Hence to obtain the same preferred Nash equilibrium [among (1,1) and (0,0)] for both the diners $D_1$ and $D_2$, we take $|a|=1/\sqrt{2}$ which makes the payoff (for both the diners) the same at both the equilibrium points, i.e., $\$_{1,2}(1,1)=\$_{1,2}(0,0)=(u_2+u_1)/2$. It can easily be verified, for the third Nash equilibrium strategy, that $\$_1(\bar{p}_1,\bar{p}_2)=\$_2(\bar{p}_1,\bar{p}_2)\le 3(u_1+u_2)/8<(u_1+u_2)/2$. Thus the entangled state

$$|\psi_{in}\rangle = \frac{|R_1R_2\rangle-|R_2R_1\rangle}{\sqrt{2}}, \tag{A17}$$

corresponds to the best Nash equilibrium. It can also be argued from the symmetry of the payoff for the two diners, as shown in Table II, that one expects the best Nash equilibrium to occur when $|a|=1/\sqrt{2}$ in Eq. (A2). The above argument can be extended to the case of $N$ diners and $N$ restaurants each with a different ranking[10] and one can deduce that the best Nash equilibrium strategy corresponds to the many-particle entangled state (i.e., the integer quantum Hall state at filling factor 1) chosen by us.


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